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## NAKAYAMA ISOMORPHISMS FOR THE MAXIMAL QUOTIENT RING OF A LEFT HARADA RING

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## **Abstract**

From several results of Kado and Oshiro, we see that if the maximal quotient ring of a given left Harada ring  $R$  of type  $(*)$  has a Nakayama automorphism, then  $R$  has a Nakayama isomorphism. This result poses a question whether if the maximal quotient ring of a given left Harada ring  $R$  has a Nakayama isomorphism, then  $R$  has a Nakayama isomorphism. In this paper, we shall show that a basic ring of the maximal quotient ring of a given Harada ring has a Nakayama isomorphism if and only if its Harada ring has a Nakayama isomorphism.

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Dedicated to Professor Takeshi Sumioka on the Occasion of His Sixtieth Birthday

KAZUAKI NONOMURA

**ABSTRACT.** From several results of Kado and Oshiro, we see that if the maximal quotient ring of a given left Harada ring  $R$  of type  $(*)$  has a Nakayama automorphism, then  $R$  has a Nakayama isomorphism. This result poses a question whether if the maximal quotient ring of a given left Harada ring  $R$  has a Nakayama isomorphism, then  $R$  has a Nakayama isomorphism. In this paper, we shall show that a basic ring of the maximal quotient ring of a given Harada ring has a Nakayama isomorphism if and only if its Harada ring has a Nakayama isomorphism.

### 1. INTRODUCTION

Let  $R$  be a basic left Harada ring. Then we have a complete set

$$\{e_{11}, \dots, e_{1n(1)}, \dots, e_{m1}, \dots, e_{mn(m)}\}$$

of primitive idempotents for  $R$  such that for each  $i = 1, \dots, m$

- (a)  $e_{i1}R$  is injective as a right  $R$ -module;
- (b)  $J(e_{i,k-1}R) \cong e_{ik}R$  for each  $k = 2, \dots, n(i)$ .

We call  $R$  a left Harada ring of type  $(*)$  if there exists an unique  $g_i$  in  $\{e_{in(i)}\}_{i=1}^m$  for each  $i = 1, \dots, m$  such that the socle of  $e_{i1}R$  is isomorphic to  $g_iR/J(g_iR)$  and the socle of  $Rg_i$  is isomorphic to  $Re_{i1}/J(Re_{i1})$ .

Oshiro [9] showed the following;

**Result A** ([9, Theorem 2]). *Suppose that  $R$  is a left Harada ring which is not of type  $(*)$ . Then there exists a series of left Harada rings  $T_1, \dots, T_n$  and surjective ring homomorphisms  $\phi_1, \dots, \phi_n$ :*

$$T_1 \xrightarrow{\phi_1} T_2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_{n-1}} T_n \xrightarrow{\phi_n} R$$

such that

- (1)  $T_1$  is of type  $(*)$ , and
- (2)  $\text{Ker } \phi_i$  is a simple ideal of  $T_i$  for any  $i \in \{1, \dots, n\}$ .

Kado and Oshiro [7] showed the following results;

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*Key words and phrases.* Maximal quotient rings; Harada rings; Nakayama isomorphisms.

**Result B** ([7, Proposition 5.3]). *If every basic QF rings has a Nakayama automorphism, then every basic left Harada ring of type  $(*)$  has a Nakayama isomorphism.*

**Result C** ([7, Proposition 5.4]). *Let  $S$  be a two-sided ideal of  $R$  that is simple as a left ideal and as a right ideal. If  $R$  has a Nakayama isomorphism, then  $R/S$  has a Nakayama isomorphism.*

Moreover Kado showed the following;

**Result D** ([6, Corollary]). *The maximal quotient ring of a left Harada ring of type  $(*)$  is a QF ring.*

Using these four results, we see that if the maximal quotient ring of a given left Harada ring  $R$  of type  $(*)$  has a Nakayama automorphism, then  $R$  has a Nakayama isomorphism. So this statement poses a question whether if the maximal quotient ring of a given left Harada ring  $R$  has a Nakayama isomorphism, then  $R$  has a Nakayama isomorphism. In this paper, we shall show that the maximal quotient ring of a given left Harada ring  $R$  has a Nakayama isomorphism iff  $R$  has a Nakayama isomorphism.

Throughout this paper, we assume that all rings are associative rings with identity and all modules are unitary. We denote the set of primitive idempotents for  $R$  by  $\text{Pi}(R)$ , and denote a complete set of primitive idempotents for  $R$  by  $\text{pi}(R)$ . By  $M_R$  (resp.  ${}_R M$ ), we mean that  $M$  is a right (resp. left)  $R$ -module. For a module  $M$ , we denote the Jacobson radical of  $M$  by  $J(M)$ , the injective hull of  $M$  by  $E(M)$ , the socle of  $M$  by  $S(M)$ , respectively.  $L \leq M$  (resp.  $L < M$ ) means  $L$  is a submodule of  $M$  (resp.  $L \leq M$  and  $L \neq M$ ).

We call a one-sided artinian ring  $R$  right (resp. left) QF-3 ring if  $E(R_R)$  (resp.  $E({}_R R)$ ) is projective, respectively.

We denote the maximal left (resp. right) quotient ring of  $R$  by  $Q_\ell(R)$  (resp.  $Q_r(R)$ ), respectively, and denote the maximal left and maximal right quotient ring of  $R$  by  $Q(R)$ . If a ring is QF-3, its maximal left quotient ring and its right quotient ring coincide by [12, Theorem 1.4].

## 2. Maximal quotient ring

We list some basic results, which several authors showed, for our main result in this paper. Recall that for  $e, f \in \text{Pi}(R)$ , we say that the pair  $(eR : Rf)$  is an  $i$ -pair if  $S(eR) \cong fR/J(fR)$  and  $S(Rf) \cong Re/J(Re)$ .

**Lemma 2.1** ([5]). *Let  $R$  be a one-sided artinian ring, and let  $e \in \text{Pi}(R)$ . Then the following conditions are equivalent:*

- (1)  $eR$  is injective as a right  $R$ -module.
- (2) There exists some  $f \in \text{Pi}(R)$  such that  $(eR : Rf)$  is an  $i$ -pair.

In this case,  $Rf$  is also injective as a left  $R$ -module.

Let  $R$  be a left perfect ring. Then  $R$  has a primitive idempotent  $e$  with  $S(R_R)e \neq 0$ . If  $R$  is QF-3, then the primitive idempotent  $e$  with  $S(R_R)e \neq 0$  are characterized as follows;

**Lemma 2.2** ([4, Theorem 2.1]). *Let  $R$  be a one-sided artinian QF-3 ring, and let  $e \in \text{Pi}(R)$ . Then  ${}_R R e$  is injective if and only if  $S(R_R)e \neq 0$ .*

We call  $e \in \text{Pi}(R)$  right (resp. left)  $S$ -primitive if  $S(R_R)e \neq 0$  (resp.  ${}_e S(R_R) \neq 0$ ), respectively.

The following statement, which Storrer [11, Proposition 4.8] showed, is helpful in this paper.

**Lemma 2.3** ([11, Proposition 4.8]). *Let  $R$  and  $Q = Q(R)$  be left perfect. Then*

- (1) *If  $e$  is a right  $S$ -primitive idempotent for  $R$ , then so is it for  $Q$ .*
- (2) *If  $e_1, e_2$  are right  $S$ -primitive idempotents for  $R$ , then  $e_1 R \cong e_2 R$  if and only if  $e_1 Q \cong e_2 Q$ .*
- (3) *If  $e$  is a right  $S$ -primitive idempotent for  $Q$ , then there exists a right  $S$ -primitive idempotent  $e' \in R$  such that  $eQ \cong e'Q$ .*

A ring  $R$  is called a left Harada ring if it is left artinian and its complete set  $\text{pi}(R)$  of orthogonal primitive idempotents is arranged as follows:

$$\text{pi}(R) = \bigcup_{i=1}^m \{e_{ij}\}_{j=1}^{n(i)},$$

where

- (a) each  $e_{i1}R_R$  is an injective module for each  $i = 1, 2, \dots, m$ .
- (b)  $e_{i,k-1}R_R \cong e_{ik}R$ , or  $J(e_{i,k-1}R_R) \cong e_{ik}R$  for each  $i$  and each  $k = 2, 3, \dots, n(i)$ .
- (c)  $e_{ik}R \not\cong e_{jt}R$  for  $i \neq j$ .

*Remark.* Let  $R$  be a left Harada ring. Then  $Q(R)$  is also a left Harada ring (See [6, Theorem 4]) and a complete set  $\text{pi}(R)$  of orthogonal primitive idempotents for  $R$  is also the one of  $Q$  (See [6, p.248]).

Using Remark 2, Kado showed the following;

**Proposition 2.4** ([6, Proposition 2]). *Let  $R$  be a left Harada ring, and let  $(eR : Rf)$  be an  $i$ -pair for  $e, f \in \text{pi}(R)$ . Then  $(eQ(R) : Q(R)f)$  is an  $i$ -pair*

*Remark.* Let  $R$  be a basic and left Harada ring. Then we have a complete set of orthogonal primitive idempotents  $\text{pi}(R) = \bigcup_{i=1}^m \{e_{ij}\}_{j=1}^{n(i)}$  for  $R$  satisfying the following conditions:

- (a)  $e_{i1}R_R$  is injective for each  $i = 1, \dots, m$ ,
- (b)  $e_{i,j+1}R_R \cong J(e_iR_R)$  for each  $j = 1, \dots, n(i) - 1$ .

We have a complete set  $\{Rg_1, \dots, Rg_m\}$  of pairwise non-isomorphic indecomposable injective projective left  $R$ -modules, such that the  $(e_{i1}R : Rg_i)$  are  $i$ -pair for each  $i = 1, \dots, m$  since  $R$  is basic and artinian QF-3. So the number of right  $S$ -primitive is  $m$  by Lemma 2.2.

Recall the following notation [6, p.249]. Let  $\theta : fR \rightarrow eR$  be an  $R$ -monomorphism such that  $\text{Im } \theta = J(eR)$ , where  $e, f \in \text{Pi}(R)$ . Then by [11, Proposition 4.3],  $\theta$  can be uniquely extended to a  $Q_r(R)$ -homomorphism  $\theta^* : fQ_r(R) \rightarrow eQ_r(R)$ .

We shall need the following results.

**Lemma 2.5** ([6, Proposition 3]). *Let  $R$  be a basic and left Harada ring, and  $Q = Q(R)$  and  $\theta$  as above. Then the following hold.*

- (1) *If  $e$  is not right  $S$ -primitive, then the extension  $\theta^* : fQ \rightarrow eQ$  is an isomorphism.*
- (2) *If  $e$  is right  $S$ -primitive, then the extension  $\theta^* : fQ \rightarrow eQ$  is a monomorphism such that  $\text{Im } \theta^* = J(eQ)$ .*

*Remark* (cf. [11, Lemma 4.2]). Let  $\{g_i\} \cup \{f_j\}$  be a complete set of orthogonal primitive idempotents for  $R$ , where the  $g_i$  are right  $S$ -primitive and the  $f_j$  are not right  $S$ -primitive. We denote  $g_0$  by  $g_0 = \sum g_i$ . Then  $Q(R)g = Rg$  and  $Q(R)g_0 = Rg_0$  for every right  $S$ -primitive idempotent  $g$  of  $R$ .

Let  $R$  be a basic left artinian ring, and let  $\{e_1, e_2, \dots, e_n\}$  be a complete set of orthogonal primitive idempotents for  $R$  and let

$$S = \text{End}_R(\oplus_{i=1}^n E(Re_i/J(Re_i)))$$

be the endomorphism ring of a minimal injective cogenerator for the category of left  $R$ -modules. Let  $f_i$  be the primitive idempotent for  $S$  corresponding to the projection

$$\oplus_{i=1}^n E(Re_i/J(Re_i)) \rightarrow E(Re_i/J(Re_i)).$$

Then we call a ring isomorphism  $\tau : R \rightarrow S$  a Nakayama isomorphism if  $\tau(e_i) = f_i$  for each  $i = 1, 2, \dots, n$ . By [3, p.42], the existence of a Nakayama isomorphism does not depend on the choice of the complete set  $\{e_1, e_2, \dots, e_n\}$  of orthogonal primitive idempotents. (See [7, Remark on p.387].)

It is important whether the maximal quotient ring of a basic artinian ring is basic since a Nakayama isomorphism is defined on a basic ring. Here we shall study the case that the maximal quotient ring of a given left Harada ring is basic.

**Theorem 2.6** (cf. [2, Corollary 22]). *Let  $R$  be a basic and left Harada ring and  $Q = Q(R)$ . Then  $Q$  is a basic ring if and only if  $R$  either is QF or satisfies the following;  $n(i) = 1$  or  $2$  and  ${}_R Re_{i1}$  is injective for any  $i$ . In this case  $R = Q$ .*

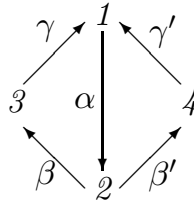
*Proof.* Note that both  $R$  and  $Q$  are artinian QF-3. Assume that  $Q$  is basic. Let  $e_{i,k+1}, e_{ik} \in \{e_{ij}\}_{j=2}^{n(i)}$ . Then we have an  $R$ -monomorphism  $\theta_{ik} : e_{i,k+1}R \rightarrow e_{ik}R$  such that  $\text{Im } \theta_{ik} = J(e_{ik}R)$ . If  $e_{ik}$  is not right  $S$ -primitive, then  $e_{i,k+1}Q \cong e_{ik}Q$  by Lemma 2.5. This contradicts that  $Q$  is basic. Hence  $e_{ik}$  is right  $S$ -primitive for  $k = 1, 2, \dots, n(i) - 1$ . Since the  $Re_{ik}$  are injective for each  $k = 1, 2, \dots, n(i) - 1$  by Lemma 2.2, there exists some  $Rg$  in  $\{Rg_1, \dots, Rg_m\}$  such that  $Re_{ik} \cong Rg$ . However  $R$  is basic, so we see that  $n(i) = 1$  or  $2$  and  $e_{i1}$  is right  $S$ -primitive.

In case  $n(i) = 1$  for every  $i = 1, \dots, m$ , then  $R$  is QF.

In case  $n(i) = 2$  for some  $i \in \{1, \dots, m\}$ . If  $e_{in(i)}$  is right  $S$ -primitive, then  ${}_R Re_{in(i)}$  is injective by Lemma 2.2. Hence  $e_{in(i)}$  is not right  $S$ -primitive since  ${}_R Re_{i1}$  is injective and so  $\{Rg_1, \dots, Rg_m\} = \{Re_{11}, \dots, Re_{m1}\}$ .

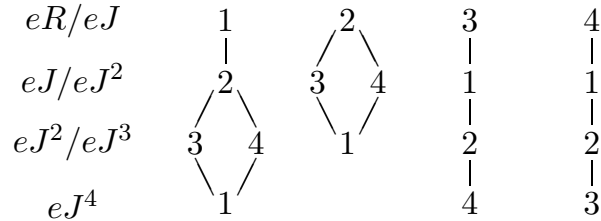
Conversely, first, assume that  $R$  is QF. Since  ${}_R Re$  is injective for any  $e \in \text{pi}(R)$ ,  $e$  is right  $S$ -primitive by Lemma 2.2. Thus,  $eQ \not\cong fQ$  for any  $e, f \in \text{pi}(R) = \text{pi}(Q)$  by Lemma 2.3. Therefore  $Q$  is basic. Next, assume that  $R$  satisfies  $n(i) = 1$  or  $2$  and  $Re_{i1}$  is injective for any  $i$ . Then  $e_{i1}$  is left  $S$ -primitive and so  $eQ = eR$  by Remark 2. Hence  $J(eQ) = J(eR)$ . Therefore it is also clear to see that  $R = Q$ .  $\square$

**Example.** We shall give a basic left Harada ring  $R$  with  $J(R)^5 = 0$ , which is not QF. Let  $R$  be an algebra over a field  $K$  defined by the following quiver;



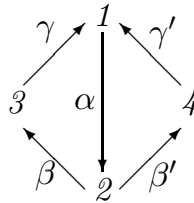
with the relations  $\gamma\beta = \gamma'\beta'$ ,  $\alpha\gamma\beta = 0$ , and  $\beta'\alpha\gamma = 0$ .

The composition diagrams of the Loewy factors of the indecomposable projective modules of  $R_R$  is the following.

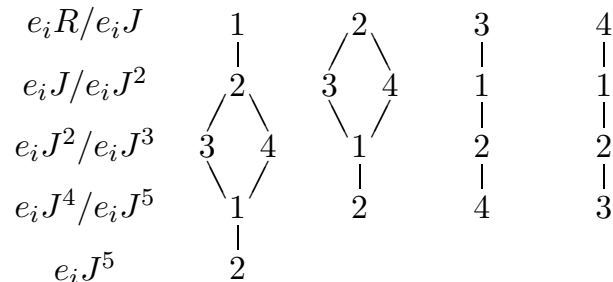


Then  $R$  is a left Harada ring which is not  $QF$  since  $e_1R_R$ ,  $e_3R_R$  and  $e_4R_R$  are injective and  $e_2R_R \cong J(e_1R)$ . Moreover  $e_1, e_3, e_4$  are right  $S$ -primitive. Hence  $e_1Q(R) = e_1R$ ,  $e_3Q(R) = e_3R$  and  $e_4Q(R) = e_4R$  are injective and  $e_2Q(R) \cong J(e_1Q(R))$ . Therefore  $R = Q(R)$ .

**Example.** We shall give a basic Harada ring  $R$  with  $J(R)^6 = 0$ , but  $Q(R)$  is not basic. Let  $R$  be an algebra over a field  $K$  defined by the following quiver;



with the relations  $0 = \beta\alpha\gamma\beta = \beta'\alpha\gamma'\beta' = \beta\alpha\gamma = \beta'\alpha\gamma'$ , and  $\gamma\beta = \gamma'\beta'$ . Then the composition diagrams of the Loewy factors of the indecomposable projective modules of  $R_R$  is the following.



Then since  $e_1R_R$ ,  $e_3R_R$  and  $e_4R_R$  are injective and  $e_2R_R \cong J(e_1R)$ ,  $R$  is a left Harada ring which is not  $QF$ . Hence  $e_2Q(R) \cong e_1Q(R)$  since  $e_1$  is not right  $S$ -primitive. Therefore  $Q(R)$  is not basic.

### 3. Nakayama isomorphism

In this section, we study the Nakayama isomorphisms for the representative matrix ring of a basic left Harada ring and its maximal quotient ring. Let  $R$  be a basic left Harada ring, and let  $\text{pi}(R) = \bigcup_{i=1}^m \{e_{ij}\}_{j=1}^{n(i)}$  be a complete set of orthogonal primitive idempotents as in Remark 2. Furthermore,



let  $R^*$  be the representative matrix ring of  $R$ .  $R^*$  is represented as block matrices as follows:

$$R^* = \begin{pmatrix} R_{11}^* & \cdots & R_{1m}^* \\ \vdots & \ddots & \vdots \\ R_{m1}^* & \cdots & R_{mm}^* \end{pmatrix},$$

where  $R_{ij}^* = P_{ij}$  for  $j \neq \sigma(i)$  and  $R_{i\sigma(i)}^* = P_{i\sigma(i)}^*$  (See [7, Section 4]).

Here, adding one row and one column to  $R^*$ , we make an extended matrix ring  $W_i(R)$  of  $R$  as follows:

$$\begin{pmatrix} R_{11}^* & \cdots & \cdots & R_{1i}^* & Y_1 & R_{1,i+1}^* & \cdots & R_{1m}^* \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ R_{i1}^* & \cdots & \cdots & R_{ii}^* & Y_i & R_{i,i+1}^* & \cdots & R_{im}^* \\ X_1 & \cdots & X_{i-1} & X_i & Q & X_{i+1} & \cdots & X_m \\ R_{i+1,1}^* & \cdots & \cdots & R_{i+1,i}^* & Y_{i+1} & R_{i+1,i+1}^* & \cdots & R_{i+1,m}^* \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ R_{m1}^* & \cdots & \cdots & R_{mi}^* & Y_m & R_{m,i+1}^* & \cdots & R_{mm}^* \end{pmatrix},$$

where  $X_k$  is the last row of  $R_{ik}^*$  ( $k = 1, \dots, m$ ,  $k \neq i$ ),  $Y_k$  is the last column of  $R_{ki}^*$  ( $k = 1, \dots, m$ ),  $X_i = (P_{in(i),i1}^* \cdots P_{in(i),in(i)-1}^* J(P_{in(i),in(i)}^*))$ , and  $Q = P_{in(i),in(i)}^*$ .

Then  $W_i(R)$  naturally becomes a ring by operations of  $R^*$ . We call this the  $i$ -th extended ring of  $R$ .

**Proposition 3.1** ([7, Proposition 5.11]). *If  $W_i(R)$  has a Nakayama isomorphism, then  $R$  also has a Nakayama isomorphism.*

Let  $R$  be a basic and left Harada ring, and let

$$\text{pi}(R) = \bigcup_{i=1}^m \{e_{ij}\}_{j=1}^{n(i)}$$

be a complete set of orthogonal primitive idempotents for  $R$  as given in Remark 2. Then (See [7, p.388]), for any  $e_{ij}$  in  $\text{pi}(R)$ , there exists some  $g_i$  in  $\text{pi}(R)$  with  $Rg_i$  injective such that  $E(Re_{ij}/J(Re_{ij})) \cong Rg_i/S_{j-1}(Rg_i)$ , where  $S_j(Rg_i)$  is the  $j$ -th socle of  $Rg_i$ . We denote the generator  $g_i + S_{j-1}(Rg_i)$  of  $Rg_i/S_{j-1}(Rg_i)$  by  $g_{ij}$  for each  $i = 1, \dots, m$ ,  $j = 1, \dots, n(i)$ . By [7, Proposition 3.2], a minimal injective cogenerator  $G = \oplus_{i,j} Rg_{ij}$  is finitely generated. Therefore we note that  $R$  is left Morita dual to  $\text{End}_R(G)$  by [1, Theorem 30.4]. We call this  $\text{End}({}_R G)$  the *dual ring* of  $R$ . We denote the dual ring of  $R$  by  $T(R)$ .

For the proof of proposition 3.2 below, we denote

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & R_{ij}^* & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} \subseteq R^*$$

by  $[R_{ij}^*]$  and

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & R_{ij}^* & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} \subseteq W_i(R)$$

by  $[R_{ij}^*]^w$ ,

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & X_k & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} \subseteq W_i(R)$$

by  $[X_k]^w$ ,

$$\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & Y_l & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \subseteq W_i(R)$$

by  $[Y_l]^w$ ,

$$\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & Q & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \subseteq W_i(R)$$

by  $[Q]^w$ .

By using the result that Kado and Oshiro [7, Proposition 5.11] showed, we shall show the following proposition. The proposition is essential in this paper.

**Proposition 3.2.**  *$W_i(R)$  has a Nakayama isomorphism if and only if so does  $R$ .*

*Proof.* ( $\Rightarrow$ ). By Proposition 3.1 ([7, Proposition 5.11]). ( $\Leftarrow$ ). As [7, Proposition 5.11], let  $e_{ij}$  be the matrix of  $R^*$  such that the  $(ij, ij)$ -component is the unity and other components are zero, and let  $w_{ij}$  be the matrix of  $W_i(R)$  such that the  $(ij, ij)$ -component is the unity and other components are zero. Note that the size of the columns in  $W_i(R)$  is  $n(i) + 1$ . Let  $\Psi$  be the natural

embedding homomorphism;

$$\begin{pmatrix} R_{11}^* & \cdots & R_{1m}^* \\ & \cdots & \\ R_{m1}^* & \cdots & R_{mm}^* \end{pmatrix} \downarrow \Psi$$

$$\begin{pmatrix} R_{11}^* & \cdots & \cdots & R_{1i}^* & 0 & R_{1,i+1}^* & \cdots & R_{1m}^* \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ R_{i1}^* & \cdots & \cdots & R_{ii}^* & 0 & R_{i,i+1}^* & \cdots & R_{im}^* \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ R_{i+1,1}^* & \cdots & \cdots & R_{i+1,i}^* & 0 & R_{i+1,i+1}^* & \cdots & R_{i+1,m}^* \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ R_{m1}^* & \cdots & \cdots & R_{mi}^* & 0 & R_{m,i+1}^* & \cdots & R_{mm}^* \end{pmatrix},$$

$\underbrace{\hspace{1.5cm}}_{i+1}$

where  $R_{ij}^* \rightarrow R_{ij}^*$  are identity maps for all  $i, j$ . Moreover let  $h_{ij}$  be the matrix of  $T(R)$  such that the  $(ij, ij)$ -component is the unity and other components are zero, and let  $v_{ij}$  be the matrix of  $W_i(T(R))$  such that the  $(ij, ij)$ -component is the unity and other components are zero. Note that the size of the columns in  $W_i(T(R))$  is  $n(i) + 1$ . Let

$$\begin{pmatrix} T(R)_{11} & \cdots & T(R)_{1m} \\ & \cdots & \\ T(R)_{m1} & \cdots & T(R)_{mm} \end{pmatrix}$$

be the representative matrix ring  $T(R)^*$  of  $T(R)$ , and let  $T(W_i(R))$  be the dual ring of  $W_i(R)$  as follows;

$$\begin{pmatrix} T(R)_{11} & \cdots & T(R)_{1i} & {}^tY_1 & T(R)_{1,i+1} & \cdots & T(R)_{1m} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ T(R)_{i1} & \cdots & T(R)_{ii} & {}^tY_i & T(R)_{i,i+1} & \cdots & T(R)_{im} \\ {}^tX_1 & \cdots & {}^tX_i & {}^tQ & {}^tX_{i+1} & \cdots & {}^tX_m \\ T(R)_{i+1,1} & \cdots & T(R)_{i+1,i} & {}^tY_{i+1} & T(R)_{i+1,i+1} & \cdots & T(R)_{i+1,m} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ T(R)_{m1} & \cdots & T(R)_{mi} & {}^tY_m & T(R)_{m,i+1} & \cdots & T(R)_{mm} \end{pmatrix}.$$

Let  $\Psi_{T(R)}$  be the natural embedding homomorphism;

$$\begin{pmatrix} T(R)_{11} & \cdots & T(R)_{1m} \\ & \cdots & \\ T(R)_{m1} & \cdots & T(R)_{mm} \end{pmatrix} \downarrow \Psi_{T(R)}$$

$$\begin{pmatrix} T(R)_{11} & \cdots & \cdots & T(R)_{1i} & 0 & T(R)_{1,i+1} & \cdots & T(R)_{1m} \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ T(R)_{i1} & \cdots & \cdots & T(R)_{ii} & 0 & T(R)_{i,i+1} & \cdots & T(R)_{im} \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ T(R)_{i+1,1} & \cdots & \cdots & T(R)_{i+1,i} & 0 & T(R)_{i+1,i+1} & \cdots & T(R)_{i+1,m} \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ T(R)_{m1} & \cdots & \cdots & T(R)_{mi} & 0 & T(R)_{m,i+1} & \cdots & T(R)_{mm} \end{pmatrix},$$

$\underbrace{\quad}_{i+1}$

where  $T(R)_{ij} \rightarrow T(R)_{ij}$  are identity maps for all  $i, j$ . We note that  $T(W_i(R)) = W_i(T(R))$  (See [7, Proposition 5.11]).

Assume that  $\varphi : R^* \rightarrow T(R)^*$  is a Nakayama isomorphism with  $\varphi(e_{ij}) = h_{ij}$ . (i.e.,  $\varphi([r_{kl}]) \in [T(R)_{kl}]$  for any  $[r_{kl}] \in [R_{ij}^*]$ , where  $(k, l)$ -componentwise of  $R_{ij}^*$  corresponds to  $(k, l)$ -componentwise of  $T(R)_{ij}$ .) We consider the following diagram;

$$\begin{array}{ccc} W_i(R) & & W_i(T(R)) \\ \uparrow \Psi & & \uparrow \Psi_{T(R)} \\ R^* & \xrightarrow{\varphi} & T(R)^*. \end{array}$$

Here we define a map  $\bar{\varphi} : W_i(R) \rightarrow W_i(T(R))$  as follows;

- (a)  $\bar{\varphi}([r_{kl}]^w) = [\varphi([r_{kl}])]^w \in [T(R)_{kl}]^w$  for any  $[r_{kl}]^w \in [R_{kl}^*]^w; 1 \leq k \leq m, 1 \leq l \leq m$ ;
- (b)  $\bar{\varphi}([x]^w) \in [{}^tX_k]^w$  for any  $[x]^w \in [X_k]; k = 1, \dots, m$ ;
- (c)  $\bar{\varphi}([y]^w) \in [{}^tY_l]^w$  for any  $[y]^w \in [Y_l]^w; l = 1, \dots, m$ ;
- (d)  $\bar{\varphi}([q]^w) \in [{}^tQ]^w$  for any  $[q]^w \in [Q]^w$ .

Since  $\varphi(e_{ij}) = h_{ij}$ ,  $\bar{\varphi}$  is well-defined. Moreover it is satisfied  $\bar{\varphi}(w_{i,n(i)+1}) = v_{i,n(i)+1}$ .  $[r_{kl}]^w \in [R_{kl}^*]^w$  implies  $[r_{kl}] \in [R_{kl}^*]$ . So we can easily check that  $\bar{\varphi}$  is a ring homomorphism. Then since  $\varphi$  is a Nakayama isomorphism, we see that  $\bar{\varphi}$  is also injective and surjective. Therefore  $\bar{\varphi}$  is a Nakayama isomorphism.  $\square$

*Remark.* We shall define a special case of an extended ring for a given ring  $R$ . Let  $\{e_1, e_2, \dots, e_n\}$  be a complete set of orthogonal primitive idempotents

for  $R$ . Then for a primitive idempotent  $e_i$  in  $R$ , we define  $R_{e_i}$  as follows;

$$\begin{pmatrix} e_1 R e_1 & \cdots & e_1 R e_i & Y_1 & e_1 R e_{i+1} & \cdots & e_1 R e_n \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \\ e_i R e_1 & \cdots & e_i R e_i & Y_i & e_i R e_{i+1} & \cdots & e_i R e_n \\ X_1 & \cdots & X_i & U & X_{i+1} & \cdots & X_n \\ e_{i+1} R e_1 & \cdots & e_{i+1} R e_i & Y_{i+1} & e_{i+1} R e_{i+1} & \cdots & e_{i+1} R e_n \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \\ e_n R e_1 & \cdots & e_n R e_i & Y_n & e_n R e_{i+1} & \cdots & e_n R e_n \end{pmatrix},$$

where the  $X_j$  are  $e_i R e_j$  for  $j = 1, \dots, i-1, i+1, \dots, n$ ,  $X_i$  is  $J(e_i R e_i)$ , the  $Y_k$  are  $e_k R e_i$  for  $k = 1, \dots, n$  and  $U$  is  $e_i R e_i$ . Then  $R_{e_i}$  is a ring by usual matrix operations.

*Remark.* Proposition 3.2 says that a basic left Harada ring  $R$  has a Nakayama isomorphism if and only if so does  $R_e$  for  $e \in \text{pi}(R) = \bigcup_{i=1}^m \{e_{ij}\}_{j=1}^{n(i)}$ .

*Remark.* If  $R$  is a one-sided artinian QF-3 ring, the number of right  $S$ -primitive idempotents for  $R$  coincides with that of left  $S$ -primitive idempotents for  $R$ .

We denote a basic ring of  $Q(R)$  by  $Q^b(R)$ .

Let  $R$  be a basic and left Harada ring, let  $Q = Q(R)$ , and let  $\text{pi}(R) = \bigcup_{i=1}^m \{e_{ij}\}_{j=1}^{n(i)}$  be a complete set of primitive idempotents for  $R$  as given in Remark 2.

(a). First, we consider the following three cases.

(i). We take  $\{e_{ij}\}_{j=1}^{n(i)}$  without right  $S$ -primitive idempotents. Then  $e_{i1}Q \cong e_{ij}Q$  for  $j = 2, \dots, n(i)$  by Lemma 2.5. So  $Q^b$  has  $e_{i1}$  as a primitive idempotent. Note that if we have  $\{e_{ij}\}_{j=1}^{n(i)}$  without right  $S$ -primitive idempotents, there exists some  $k \neq i \in \{1, \dots, m\}$  such that  $\{e_{kj}\}_{j=1}^{n(k)}$  has two or more right  $S$ -primitive idempotents by Remark 3.

(ii). We take  $\{e_{ij}\}_{j=1}^{n(i)}$  with a right  $S$ -primitive idempotent. Let  $e_{ik}$  be a right  $S$ -primitive idempotent. Then by Lemma 2.5 it is satisfied the following:

$$\begin{cases} e_{i1}Q \cong e_{ij}Q & \text{for } j = 2, \dots, k; \\ e_{i,k+1}Q \cong J(e_{ik}Q) & \text{and} \\ e_{i,k+1}Q \cong e_{ij}Q & \text{for } j = k+2, \dots, n(i). \end{cases}$$

So  $Q^b$  has  $e_{i1}, e_{ik}$  as primitive idempotents. Note that if  $e_{in(i)}$  is a right  $S$ -primitive idempotent, then  $e_{i1}Q \cong e_{ij}Q$  for  $j = 2, \dots, n(i)$  by Lemma 2.5.

(iii). We take  $\{e_{ij}\}_{j=1}^{n(i)}$  with two or more right  $S$ -primitive idempotents. Let  $e_{ikt}$  ( $2 \leq t < n(i)$ ) be right  $S$ -primitive idempotents. Then by Lemma

2.5 it is satisfied the following sequence:

$$\begin{array}{ccccccc}
 e_{i1}Q & >& e_{i1}J(Q) & & & & \\
 & & \wr \uparrow & & & & \\
 & & e_{i,k_1+1}Q & >& J(e_{i,k_1+1}Q) & & \\
 & & & & \wr \uparrow & & \\
 & & & & e_{i,k_2+1}Q & >& J(e_{i,k_2+1}Q) \\
 & & & & & & \wr \uparrow \\
 & & & & & & e_{i,k_3+1}Q \quad \cdots
 \end{array}$$

So  $Q^b$  has  $e_{i1}, e_{ik_t+1}$  as primitive idempotents.

Note that if every  $\{e_{ij}\}_{j=1}^{n(i)}$  for any  $i = 1, \dots, m$  has only one right  $S$ -primitive idempotent, say  $e_{ik(i)}$ , then by (ii),  $\bigcup_{i=1}^m \{e_{i1}, e_{ik(i)+1}\}$  is a complete set of the primitive idempotents  $\text{pi}(Q^b)$  for  $Q^b$  with  $e_{i1} Q^b$  is injective. Since  $e_{i1}$  is left  $S$ -primitive,  $e_{i1}R = e_{i1}Q$  by Remark 2 and so  $e_{i1}Re_{i1} = e_{i1}Qe_{i1}$ . Moreover if we have some  $i \in \{1, \dots, m\}$  such that  $\{e_{ij}\}_{i=1}^{n(i)}$  has no right  $S$ -primitive idempotents, then there exist some  $k \neq i \in \{1, \dots, m\}$  such that  $\{e_{kj}\}_{k=1}^{n(k)}$  has two or more right  $S$ -primitive idempotents by Remark 2. Let  $e = \sum_{i=1}^m e_{i1} + \sum e_{ik_t+1}$ , where the  $e_{ik_t}$  are right  $S$ -primitive. Therefore if we cooperate (i), (ii) or (iii), we can make the basic ring  $Q^b$  isomorphic to  $eRe$ . Furthermore we see that  $Q^b$  is isomorphic to  $eRe$  for some idempotent  $e$  of  $R$  if  $Q$  is not basic.

(b). Next, we consider the following three conditions:

(iv). If some  $\{e_{h1}\}_{h=1}^{n(h)} \subset \text{pi}(R)$  has the right  $S$ -primitive  $e_{hn(h)}$ , then putting  $e_h = e_{h1} + \cdots + e_{hn(h)}$ , by Lemma 2.3 and Lemma 2.5, we see  $e_hR = e_hQ$ .

(v). If  $\{e_{h1}\}_{h=1}^{n(h)}$  has no right  $S$ -primitive, then by Remark 3,  $Q_{e_{h1}}^b$  is isomorphism to a ring with the complete set  $\text{pi}(Q^b) \cup \{e_{h2}\}$  of primitive idempotents.

Let

$$Q^b = \begin{pmatrix} * & & e_{11}Re_{h1} & & * \\ & & \vdots & & \\ e_{h1}Re_{11} & \cdots & e_{h1}Re_{h1} & \cdots & e_{h1}Re_{m1} & \cdots \\ & & \vdots & & \\ * & & e_{m1}Re_{h1} & & * \\ & & \vdots & & \end{pmatrix}.$$

Then by Remark 3,

$$Q_{e_{h1}}^b = \begin{pmatrix} * & & e_{11}Re_{h1} & e_{11}Re_{h1} & & * \\ & & \vdots & \vdots & & \\ e_{h1}Re_{11} & \dots & e_{h1}Re_{h1} & e_{h1}Re_{h1} & \dots & e_{h1}Re_{m1} & \dots \\ e_{h1}Re_{11} & \dots & J(e_{h1}Re_{h1}) & e_{h1}Re_{h1} & \dots & e_{h1}Re_{m1} & \dots \\ & & \vdots & \vdots & & \\ * & & e_{m1}Re_{h1} & e_{m1}Re_{h1} & & * \\ & & \vdots & \vdots & & \end{pmatrix}.$$

For two ideals  $A, B$  of  $Q_{e_{h1}}^b$  as follows:

$$A = {}_{h1} > \begin{pmatrix} 0 & & \dots & \dots & & 0 \\ e_{h1}Re_{11} & \dots & e_{h1}Re_{h1} & e_{h1}Re_{h1} & \dots & e_{h1}Re_{m1} & \dots \\ 0 & & \dots & \dots & & 0 \end{pmatrix},$$

$$B = {}_{h1} > \begin{pmatrix} 0 & & \dots & \dots & & 0 \\ 0 & & \dots & \dots & & 0 \\ e_{h1}Re_{11} & \dots & J(e_{h1}Re_{h1}) & e_{h1}Re_{h1} & \dots & e_{h1}Re_{m1} & \dots \\ 0 & & \dots & \dots & & 0 \end{pmatrix}$$

we have  $J(A) \cong B$  by [10, Theorem 1].

Hence we have, as a ring isomorphism,

$$\begin{pmatrix} * & & e_{11}Re_{h1} & e_{11}Re_{h2} & & * \\ & & \vdots & \vdots & & \\ e_{h1}Re_{11} & \dots & e_{h1}Re_{h1} & e_{h1}Re_{h2} & \dots & e_{h1}Re_{m1} & \dots \\ e_{h2}Re_{11} & \dots & e_{h2}Re_{h1} & e_{h2}Re_{h2} & \dots & e_{h2}Re_{m1} & \dots \\ & & \vdots & \vdots & & \\ * & & e_{m1}Re_{h1} & e_{m1}Re_{h2} & & * \\ & & \vdots & \vdots & & \end{pmatrix} \cong$$

$$\begin{pmatrix} * & & e_{11}Re_{h1} & e_{11}Re_{h1} & & * \\ & & \vdots & \vdots & & \\ e_{h1}Re_{11} & \dots & e_{h1}Re_{h1} & e_{h1}Re_{h1} & \dots & e_{h1}Re_{m1} & \dots \\ e_{h1}Re_{11} & \dots & J(e_{h1}Re_{h1}) & e_{h1}Re_{h1} & \dots & e_{h1}Re_{m1} & \dots \\ & & \vdots & \vdots & & \\ * & & e_{m1}Re_{h1} & e_{m1}Re_{h1} & & * \\ & & \vdots & \vdots & & \end{pmatrix}$$

by [10, Theorem 1] again. Similarly repeating  $n(h) - 2$  times, we can make an extended ring with the complete set  $\text{pi}(Q^b) \cup \{e_{hj}\}_{j=2}^{n(h)}$  of primitive idempotents.

(vi). Assume that  $\{e_{h1}\}_{h=1}^{n(h)} \subset \text{pi}(R)$  has one or more right  $S$ -primitive idempotents. We denote a right  $S$ -primitive idempotent of  $\{e_{h1}\}_{h=1}^{n(h)}$  by  $e_{hkt}$ . We reset

$$\{e_{h1}\}_{h=1}^{n(h)} = \{e_{h1}, \dots, e_{hkt}, \dots, e_{hk_2}, \dots\}.$$

Then the complete set  $\text{pi}(Q^b)$  of  $Q^b$  is  $\bigcup_{i=1}^m \{e_{i1}, e_{i,k_t+1}\}_{t \geq 1}$ . First by the same argument above for  $e_{i1}, e_{i,k_1+1}$ , we have a ring isomorphic to a ring with the complete set  $\{e_{i1}, \dots, e_{i,k_1+1}\} \subset \text{pi}(R)$ . Next, by [10, Theorem 1], repeating the same argument like as (iv), for  $e_{i,k_1+1}, e_{i,k_2+1}$ , we have a ring isomorphism to a ring with the complete set  $\{e_{i1}, \dots, e_{ik_1}, e_{ik_1+1}, \dots, e_{ik_2}, e_{i,k_2+1}\}$ . Hence the suitable extended ring of  $Q^b$  is isomorphic to  $R$ .

Therefore by (a)-(i),(ii),(iii) and (b)-(iv),(v),(vi) above together with Proposition 3.2 (Remark 3), we get the following main theorem:

**Theorem 3.3.** *Let  $R$  be a basic and left Harada ring and let  $Q = Q(R)$ . Then  $Q$  has a Nakayama isomorphism if and only if so does  $R$ .*

**Example.** *Let*

$$V = \begin{pmatrix} Q_1 & Q_1 & Q_1 & Q_1 & A & A \\ J_1 & Q_1 & Q_1 & Q_1 & A & A \\ J_1 & J_1 & Q_1 & Q_1 & A & A \\ J_1 & J_1 & J_1 & Q_1 & A & A \\ B & B & B & B & Q_2 & Q_2 \\ B & B & B & B & J_2 & Q_2 \end{pmatrix} \quad \text{and}$$

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & S(A_{Q_2}) \\ 0 & 0 & 0 & 0 & 0 & S(A_{Q_2}) \\ 0 & 0 & 0 & 0 & 0 & S(A_{Q_2}) \\ 0 & 0 & 0 & 0 & 0 & S(A_{Q_2}) \\ 0 & 0 & S(B_{Q_1}) & S(B_{Q_1}) & 0 & 0 \\ 0 & 0 & S(B_{Q_1}) & S(B_{Q_1}) & 0 & 0 \end{pmatrix},$$



where  $Q_i$  is local, and  $J_i = J(Q_i)$  for  $i = 1, 2$ . We put  $R = V/K$ . We abbreviate this as

$$R = \begin{pmatrix} Q_1 & Q_1 & Q_1 & Q_1 & A & \overline{A} \\ J_1 & Q_1 & Q_1 & Q_1 & A & \overline{A} \\ J_1 & J_1 & Q_1 & Q_1 & A & \overline{A} \\ J_1 & J_1 & J_1 & Q_1 & A & \overline{A} \\ B & B & \overline{B} & \overline{B} & Q_2 & Q_2 \\ B & B & \overline{B} & \overline{B} & J_2 & Q_2 \end{pmatrix}.$$

Then  $R$  is a basic left Harada ring, and we have a complete set

$$\{e_{11}, e_{12}, e_{13}, e_{14}, e_{21}, e_{22}\}$$

of orthogonal primitive idempotents for  $R$ , where  $(e_{11}R; Re_{21})$  and  $(e_{21}R; Re_{12})$  are  $i$ -pairs. First, let

$$\begin{array}{ccc} & & e_{11}R \\ & & \vee \\ & e_{12}R & \rightarrow e_{11}J(R) \\ & \vee & \\ e_{13}R & \rightarrow e_{12}J(R) & \\ \vee & & \\ e_{14}R & \rightarrow e_{13}J(R) & \\ & & e_{21}R \\ & & \vee \\ & e_{22}R & \rightarrow e_{21}J(R) \end{array}$$

be projective covers. Then since  $e_{12}, e_{21}$  are right  $S$ -primitive, we have, by Lemma 2.5, the following:

$$\begin{array}{ccc} e_{12}Q(R) & \cong & e_{11}Q(R) \\ \vee & & \\ e_{14}Q(R) & \cong & e_{13}Q(R) \cong e_{12}J(Q(R)) \\ & & e_{21}Q(R) \\ & & \vee \\ e_{22}Q(R) & \rightarrow & e_{21}J(Q(R)). \end{array}$$

Hence we see

$$Q(R) \cong \begin{pmatrix} Q_1 & Q_1 & Q_1 & Q_1 & A & \overline{A} \\ Q_1 & Q_1 & Q_1 & Q_1 & A & \overline{A} \\ J_1 & J_1 & Q_1 & Q_1 & A & \overline{A} \\ J_1 & J_1 & \overline{Q_1} & \overline{Q_1} & A & \overline{A} \\ B & B & \overline{B} & \overline{B} & Q_2 & Q_2 \\ B & B & \overline{B} & \overline{B} & J_2 & Q_2 \end{pmatrix}.$$

So a basic ring of  $Q(R)$  is the following:

$$Q^b(R) \cong \begin{pmatrix} Q_1 & Q_1 & A & \overline{A} \\ J_1 & Q_1 & A & \overline{A} \\ B & \overline{B} & Q_2 & Q_2 \\ B & \overline{B} & J_2 & Q_2 \end{pmatrix}.$$

Therefore we see that, as a ring isomorphism,

$$\begin{pmatrix} Q_1 & Q_1 & A & \overline{A} \\ J_1 & Q_1 & A & \overline{A} \\ B & \overline{B} & Q_2 & Q_2 \\ B & \overline{B} & J_2 & Q_2 \end{pmatrix} \cong (e_{11} + e_{13} + e_{21} + e_{22})R(e_{11} + e_{13} + e_{21} + e_{22}).$$

$$\text{Next, adding } e_{11} \text{ to } Q^b \cong \left( \begin{array}{c|ccc} Q_1 & Q_1 & A & \overline{A} \\ J_1 & Q_1 & A & \overline{A} \\ B & \overline{B} & Q_2 & Q_2 \\ B & \overline{B} & J_2 & Q_2 \end{array} \right), \text{ according to Remark 3,}$$

$Q_{e_{11}}^b$  is isomorphic to

$$\left( \begin{array}{c|c|ccc} Q_1 & Q_1 & Q_1 & A & \overline{A} \\ J_1 & Q_1 & Q_1 & A & \overline{A} \\ \hline J_1 & J_1 & Q_1 & A & \overline{A} \\ B & B & \overline{B} & Q_2 & Q_2 \\ B & B & \overline{B} & J_2 & Q_2 \end{array} \right).$$

Then we get a ring isomorphism

$$\begin{pmatrix} Q_1 & Q_1 & Q_1 & A & \overline{A} \\ J_1 & Q_1 & Q_1 & A & \overline{A} \\ J_1 & J_1 & Q_1 & A & \overline{A} \\ B & B & \overline{B} & Q_2 & Q_2 \\ B & B & \overline{B} & J_2 & Q_2 \end{pmatrix} \cong (e_{11} + e_{12} + e_{13} + e_{21} + e_{22})R(e_{11} + e_{12} + e_{13} + e_{21} + e_{22}).$$

Moreover adding  $e_{14}$  to  $Q_{e_{11}}^b \cong \left( \begin{array}{ccc|cc} Q_1 & Q_1 & Q_1 & A & \bar{A} \\ J_1 & Q_1 & Q_1 & A & \bar{A} \\ J_1 & J_1 & Q_1 & A & \bar{A} \\ \hline B & B & \bar{B} & Q_2 & Q_2 \\ B & B & \bar{B} & J_2 & Q_2 \end{array} \right)$ , according to

Remark 3,  $(Q_{e_{11}}^b)_{e_{14}}$  is isomorphic to

$$\left( \begin{array}{ccc|c|cc} Q_1 & Q_1 & Q_1 & Q_1 & A & \bar{A} \\ J_1 & Q_1 & Q_1 & Q_1 & A & \bar{A} \\ J_1 & J_1 & Q_1 & Q_1 & A & \bar{A} \\ \hline J_1 & J_1 & J_1 & Q_1 & A & \bar{A} \\ \hline B & B & \bar{B} & \bar{B} & Q_2 & Q_2 \\ B & B & \bar{B} & \bar{B} & J_2 & Q_2 \end{array} \right) \cong R.$$

#### 4. Another question

Oshiro's result (Result A) in the introduction also poses another question whether there exist surjective ring homomorphisms  $\bar{\phi}_1, \dots, \bar{\phi}_n$  with the following commutative diagrams:

$$\begin{array}{ccccccc} Q(T_1) & \xrightarrow{\bar{\phi}_1} & Q(T_2) & \xrightarrow{\bar{\phi}_2} & \dots & \xrightarrow{\bar{\phi}_{n-1}} & Q(T_n) & \xrightarrow{\bar{\phi}_n} & Q(R) \\ \vee & & \vee & & & & \vee & & \vee \\ T_1 & \xrightarrow{\phi_1} & T_2 & \xrightarrow{\phi_2} & \dots & \xrightarrow{\phi_{n-1}} & T_n & \xrightarrow{\phi_n} & R. \end{array}$$

However K. Koike informed the author the following examples;

**Example.** Let  $Q$  be a local serial ring, and  $J(Q) \neq 0, J(Q)^2 = 0$ . Then  $J(Q) = S(Q)$ . We put

$$R = \begin{pmatrix} Q & Q \\ J & Q \end{pmatrix} / \begin{pmatrix} 0 & J \\ 0 & J \end{pmatrix},$$

where  $J = J(Q)$ . Then  $R$  is a serial ring of an admissible sequence (3,2) and so we see that  $R = Q(R)$ . Also

$$\begin{aligned} T_1 &= \begin{pmatrix} Q & Q \\ J & Q \end{pmatrix}, & T_2 &= \begin{pmatrix} Q & Q \\ J & Q \end{pmatrix} / \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix}, \\ Q(T_1) &= \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}, & Q(T_2) &= T_2. \end{aligned}$$

$\begin{pmatrix} J & J \\ J & J \end{pmatrix}$  is a unique non-trivial ideal of  $Q(T_1)$ . Hence there does not exist a surjective ring homomorphism  $Q(T_1)$  to  $Q(T_2)$ .

**Example.** We put

$$T = \begin{pmatrix} \mathbf{K} & \mathbf{K} & \mathbf{K} \\ 0 & \mathbf{K} & \mathbf{K} \\ 0 & 0 & \mathbf{K} \end{pmatrix}, I = \begin{pmatrix} 0 & 0 & \mathbf{K} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where  $\mathbf{K}$  is a field, and  $R = T/I$ . Then  $R$  is a serial ring of an admissible sequence  $(2, 2, 1)$  and we have a natural map

$$T = T_1 \rightarrow R.$$

However the maximal quotient ring  $Q(T)$  of  $T$  is the full matrix algebra with degree 3 over a field  $\mathbf{K}$  and  $Q(R) = R$ . Since  $Q(T)$  is semisimple, there does not exist a surjective ring homomorphism  $Q(T)$  to  $Q(R)$ .

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#### REFERENCES

- [1] F. W. Anderson and K. R. Fuller, "Rings and Categories of Modules," 2nd ed., Graduate Texts in Math., Vol. 13, Springer-Verlag, New York, 1992.
- [2] Y. Baba, Some classes of QF-3 Rings, *Comm. in Alg.* **28**(6), (2000), 2639-2669.
- [3] C. Faith, "Algebra II: Ring Theory", Springer-Verlag, Berlin/New York 1976.
- [4] K. R. Fuller, The structure of QF-3 rings, *Trans. Amer. Math. Soc.*, **134**, (1968), 343-354.
- [5] K. R. Fuller, On indecomposable injective over artinian rings, *Pacific J. Math.*, **29**, (1969), 115-135.
- [6] J. Kado, The maximal quotient ring of a left  $H$ -ring, *Osaka J. Math.* **27** (1990), 247-251.
- [7] J. Kado and K. Oshiro, Self-duality and Harada rings, *J. Algebra* **211** (1999), 384-408.
- [8] K. Koike, Examples of QF rings without Nakayama automorphism and  $H$ -rings without self-duality, *J. Algebra* **241** (2001), 731-744.
- [9] K. Oshiro, Lifting modules, extending modules and their applications to generalized uniserial rings, *Hokkaido Math. J.* **13** (1984), 339-346.
- [10] K. Oshiro, On Harada rings III, *Math. J. Okayama Univ.* **32** (1990), 111-118.
- [11] H. H. Storrer: Rings of quotients of perfect rings, "Math. Z." **122** (1971), 151-165.
- [12] H. Tachikawa, Quasi-Frobenius rings and generalizations. QF-3 and QF-1 rings, "Lecture Notes in Math.," Vol. 351, Berlin-New York, Springer-Verlag, 1973.

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